Robust Multigrid Preconditioner for Fast Finite Element Modeling of Microwave Devices

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***Abstract*—A robust preconditioning technique is presented for the fast finite element modeling of microwave devices. The proposed preconditioner is based on a multigrid scheme for the vector-scalar potential finite element formulation of electromagnetic problems. Numerical experiments from the application of the new preconditioner to the finite element analysis of microwave devices are used to demonstrate its superior numerical convergence and efficient memory usage.**

***Index Terms*—Finite-element method, multigrid methods, nested grids, vector and scalar potential formulation.**

I. INTRODUCTION

THE FINITE-element method (FEM) is one of the most effective and versatile techniques for the modeling of complex microwave devices [1]. Its application to practical engineering problems often results in large linear systems requiring iterative methods for their numerical solution. However, the convergence of iterative solvers tends to be unpredictable for electromagnetic wave problems, even when common preconditioners, such as incomplete LU factorization, are being used to improve convergence. The reasons for the slow convergence of the iterative solver are by now well understood and are associated with the spurious dc modes contained in the null space of the curl operator [2], [3], and the ill-conditioning of the FEM matrix resulting from the oversampling of the low-frequency physical modes [5], [6]. As it was proposed in [4], the spurious dc modes can be canceled by introducing a spurious electric charge and imposing the divergence free condition, $\nabla \cdot \vec{D} = 0$. Use of the vector-scalar potential ($\vec{A} - V$) formulation for the development of the FEM approximation is most suitable for this purpose. On the other hand, the difficulties associated with low-frequency physical modes can be tackled well using the nested multigrid method [6], [7]. More specifically, those modes that are over-sampled on the original FEM grid can be solved without loss of accuracy using a coarser grid and, subsequently, through an interpolation process, projected back onto the original grid.

The success of these remedies has prompted their combination into a nested multigrid vector-scalar potential finite element solver that was shown to exhibit outstanding convergence in conjunction with the analysis of two-dimensional electromagnetic scattering problems [7]. This paper presents the extension of this new solver to the robust FEM analysis of three-dimensional electromagnetic devices.

Consider a three-dimensional microwave device as shown in Fig. 1. The weak statement of the governing electric field vector Helmholtz equation

$$\nabla \times \nabla \times \vec{E} - \omega^2 \mu_0 \epsilon_0 \epsilon_r \vec{E} = 0 \tag{1}$$

using tangentially continuous edge elements $\vec{w}$ is well known and is given by

$$\int_\Omega \nabla \times \vec{w} \cdot \nabla \times \vec{E}\, dv - j\omega\mu_0 \int_{\partial\Omega} \hat{n} \times \vec{H} \cdot \vec{w}\, ds - \omega^2 \mu_0 \epsilon_0 \int_\Omega \vec{w} \cdot \epsilon_r \vec{E}\, dv = 0. \tag{2}$$

In order to address the general case, the computational domain Ω is assumed to be bounded by both microwave port boundaries ($S_i,\ i = 1, 2, \ldots, N$) and a numerical truncation boundary of S_0 on which a first-order absorbing boundary condition is imposed.

On each waveguide port, it is assumed that the associated modes are available for the representation of the fields. To simplify the presentation and without loss of generality, it is assumed that at each waveguide port only the fundamental mode propagates. Following standard microwave circuit analysis procedures, let S_1 be the excitation port. The weak form of the driven problem becomes

$$\int_\Omega \nabla \times \vec{w} \cdot \nabla \times \vec{E}\, dv + jk_0 \int_{S_0} \hat{n} \times \vec{w} \cdot \hat{n} \times \vec{E}\, ds + \sum_{i=1}^{N} jk_{z,i} \int_{S_i} \hat{n} \times \vec{w} \cdot \hat{n} \times \vec{E}\, ds - \omega^2 \mu_0 \epsilon_0 \int_\Omega \vec{w} \cdot \epsilon_r \vec{E}\, dv = 2jk_{z,1} \int_{S_1} \hat{n} \times \vec{w} \cdot \hat{n} \times \vec{e}_1\, ds \tag{3}$$

where $k_{z,i}$ is the propagation constant for the ith port and $\vec{e}_i$ is the normalized field distribution of the excitation mode on S_i. The associated FEM matrix form of the above weak statement is

$$M_{EE}^h x_E^h = f_E^h. \tag{4}$$

Once the numerical solution has been obtained, the reflection coefficient at port S_1 and the transmission coefficients at the remaining ports are obtained as follows

$$S_{11} = \int_{S_1} \hat{n} \times \vec{E} \cdot \hat{n} \times \vec{e}_1\, ds - 1$$
$$S_{i1} = \int_{S_i} \hat{n} \times \vec{E} \cdot \hat{n} \times \vec{e}_i\, ds. \tag{5}$$

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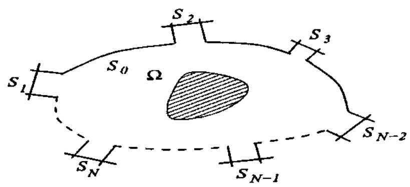

Fig. 1. Convergence behavior for various cube sizes.

The multigrid preconditioning used for the iterative solution of (4) is described next. In the iterative solution of (3), the pseduo-residual equation

$$M_{EE}^h z_E = r_E \tag{6}$$

is transformed to the matrix equation for the vector and scalar potential formulation.

As suggested in [4], the electric field is written as $\vec{E} = \vec{A} + \nabla V$. Thus the weak form of the vector wave equation (1) is

$$\begin{aligned}\int_\Omega \nabla \times \vec{w} \cdot \nabla \times \vec{A}\, dv + jk_0 \int_{S_0} \hat{n} \times \vec{w} \cdot \hat{n} \times (\vec{A} + \nabla V)\, ds \\ + \sum_i jk_{z,i} \int_{S_i} \hat{n} \times \vec{w} \cdot \hat{n} \times (\vec{A} + \nabla V)\, ds \\ - \omega^2 \mu_0 \epsilon_0 \int_\Omega \vec{w} \cdot \epsilon_r (\vec{A} + \nabla V)\, dv \\ = 2jk_{z,1} \int_{S_1} \hat{n} \times \vec{w} \cdot \hat{n} \times \vec{e}_1\, ds.\end{aligned} \tag{7}$$

The weak form of the divergence-free equation $\nabla \cdot \vec{D} = 0$ is obtained through its multiplication with the gradient of the scalar basis $\nabla\phi$

$$\begin{aligned}jk_0 \int_{S_0} \hat{n} \times \nabla\phi \cdot \hat{n} \times (\vec{A} + \nabla V)\, ds + \sum_i jk_{z,i} \int_{S_i} \hat{n} \times \nabla\phi \\ \cdot \hat{n} \times (\vec{A} + \nabla V)\, ds - \omega^2 \mu_0 \epsilon_0 \int_\Omega \nabla\phi \cdot \epsilon_r (\vec{A} + \nabla V)\, dv \\ = 2jk_{z,1} \int_{S_1} \hat{n} \times \nabla\phi \cdot \hat{n} \times \vec{e}_1\, ds.\end{aligned} \tag{8}$$

In matrix form, (7) and (8) are written as

$$\begin{pmatrix} M_{AA}^h & M_{AV}^h \\ M_{VA}^h & M_{VV}^h \end{pmatrix} \begin{pmatrix} x_A^h \\ x_V^h \end{pmatrix} = \begin{pmatrix} f_A^h \\ f_V^h \end{pmatrix} \tag{9}$$

where $M_{AA}^h = M_{EE}^h$, x_A^h and x_V^h contain the expansion coefficients for the vector and scalar potential, respectively, and the entries of the remaining matrices become evident through a direct comparison of (9) with (7) and (8).

Let Φ denote the space spanned by the scalar basis functions ϕ, and $\vec{W}$ the space spanned by the vector edge-element basis functions $\vec{w}$. $\nabla\Phi$ (Whitney-0 form) is a subset of $\vec{W}$ (Whitney-1 form) as discussed in [8]. Thus a transition matrix exists such that

$$\nabla\Phi = \vec{W} G. \tag{10}$$

Consequently, $\vec{A} - V$ matrix of (9) can be written in terms of (4)

$$\begin{pmatrix} M_{AA}^h & M_{AV}^h \\ M_{VA}^h & M_{VV}^h \end{pmatrix} = \begin{pmatrix} I \\ G^T \end{pmatrix} M_{EE}^h (I \quad G), \qquad \begin{pmatrix} f_A^h \\ f_V^h \end{pmatrix} = \begin{pmatrix} I \\ G^T \end{pmatrix} f_E^h \tag{11}$$

where I is the identity matrix [4].

This result suggests the following method for the calculation of the pseudo-residual equation of (6). First, $r_A^h \leftarrow r_E^h$. Next, the equation $M_{AA}^h z_A^h = r_A^h$ is solved for the expansion coefficients of the vector potential. This is followed by a correction step associated with the explicit imposition of the weak form of the divergence-free requirement, which involves the solution of the next equation

$$M_{VV}^h z_V^h = G^T r_A^h - M_{VA}^h z_A^h. \tag{12}$$

The approximate solution of these two equations is effected either through an incomplete Cholesky factorization or through Gauss–Seidel method. Once the approximate solution on the $\vec{A} - V$ formulation is obtained, it is transformed back to $\vec{E}$ formulation using the relationship

$$z_E^h = (I \quad G) \begin{pmatrix} z_A^h \\ z_V^h \end{pmatrix}. \tag{13}$$

The expansion of the single-level preconditioner of the pseudo-residual equation to multilevel is described in the following typical multigrid pseudo-code, MG(z_E, r_E).

1) $z_E^h \leftarrow 0$
2) if coarsest grid, then solve $M_{EE}^h z_E^h = r_E^h$
3) else
   - 3a) Relax(z_E^h, r_E^h) for v_1 times.
   - 3b) $r_E^{2h} \leftarrow I_h^{2h}(r_E^h - M_{EE}^h z_E^h)$
   - 3c) MG(z_E^{2h}, r_E^{2h})
   - 3d) $z_E^h \leftarrow z_E^h + I_{2h}^h z_E^{2h}$
   - 3e) Relax(z_E^h, r_E^h) for v_2 times.

Relax(z_E^h, r_E^h) is the single-level preconditioner which projects the problem to the $\vec{A} - V$ formulation in order to impose the divergence-free condition explicitly. I_h^{2h} is the restriction operator that maps the residual of the fine grid onto the coarser grid; I_{2h}^h is the interpolation operator that maps the correction obtained on the coarser grid onto the fine grid. In [7], it is shown I_{2h}^h is the transition matrix between the bases on the coarse and fine grids and I_h^{2h} is its transpose.

II. Numerical Results

To implement the iterative FEM solver, the proposed preconditioner is combined with the conjugate gradient (CG) method. All calculations discussed in this paper were done on a Pentium III 600 MHz PC. The stopping criterion for the iterative solver was $\|r\|_\infty = tol \cdot \|f\|_\infty$ where $tol = 10^{-4}$. For all the examples, a two-level multigrid was used, with the number of pre-smoothing and post-smoothing operations taken to be $v1 = v2 = 2$.

The low-pass filter studied in [1] is analyzed first. The dimensions of the filter are shown in the insert of Fig. 2. The first-order

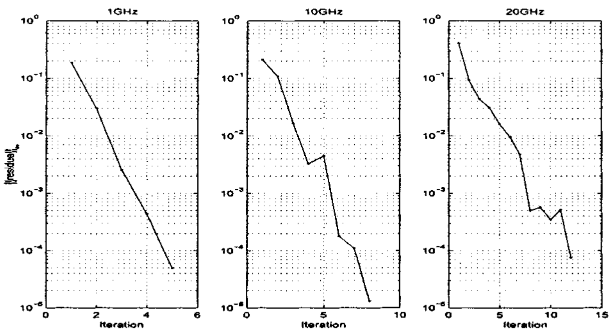

Fig. 3. Convergence of the FEM solution of the low-pass filter.

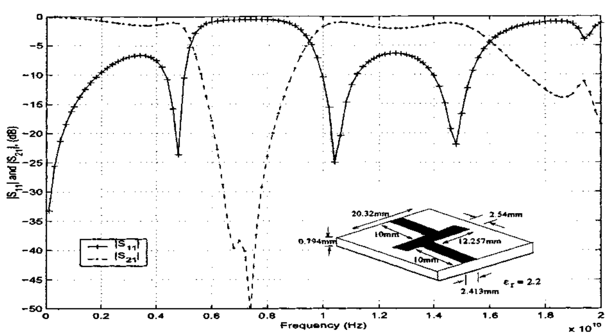

Fig. 2. Magnitude of S parameters for a low-pass filter.

absorbing boundary is placed 4mm away from the substrate and the side ends of the filter. On the finest grid, the average grid size is 0.79 mm, and the number of unknowns is 60 534. On the coarse grid, the average grid size is 1.57 mm and the number of unknowns is 7823. The calculated scattering parameters are in very good agreement with the results in [1].

The excellent convergence of the preconditioned iterative solver is demonstrated through the curves in Fig. 3. The required CPU time for the solution at 1, 10, and 20 GHz is 21.72, 27.58, and 35.21 s, respectively. The required memory is 20 MB. On the average, 10 iterations per frequency point were required over the 0.1–20 GHz bandwidth, with corresponding CPU time of 25 s. For the sake of comparison, it is mentioned that the FEM model of [1] with 33 352 unknowns required 20 min per frequency point in the lower frequency range and 10 min per frequency point for higher frequencies.

The FEM model of the annular microstrip resonator shown in the insert of Fig. 4 required a $18 \times 18 \times 2.635$ mm^3 domain wit h absorbing boundaries set 2 mm away from the structure. On the fine grid, the average grid size was 0.70 mm and the number of unknowns 27 840. On the coarse grid, the grid size was 1.4 mm and the number of unknowns 3521. The calculated scattering parameters are in excellent agreement with the measurement data in [9]. Once again, the convergence of the solver was excellent, with ten iterations per frequency and 11 seconds

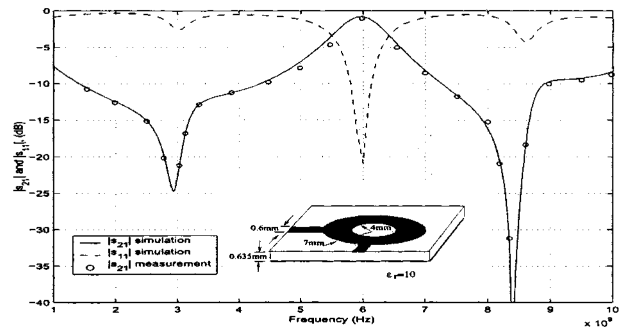

Fig. 4. Magnitude of the S parameters for an annular microstrip resonator.

of CPU time required on the average for the solution at each frequency point over the 1–10 GHz bandwidth of interest.

III. Conclusion

In conclusion, an efficient preconditioner has been proposed and demonstrated for the robust, expedient, and broadband finite element analysis of microwave devices. Through the combination the $\vec{A} - V$ formulation of the FEM approximation with the multigrid method, the ill-conditioning of the FEM matrix is avoided, and a fast converging conjugate gradient-based iterative FEM solver results. On-going studies explore the extension of the new algorithm to a broader class of electromagnetic propagation, radiation, and scattering problems.

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